Lois de comportement pour le calcul de structures sous chargement cyclique

Choix et influence sur les réponses asymptotiques

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Choices and convictions

- A plasticity course:
  - The modeling scale? Dislocations? Crystals? Polycrystals? « **Homogeneous** » materials and structures
  - The objective? A fine description of physical phenomena? **Structural computations under cyclic loadings**

- Some strong assumptions:
  - (Generally), asymptotic responses are necessary and **sufficient** for a fatigue analysis of structures
  - If possible, asymptotic response has to be **directly estimated**
  - This requires a **complete** and **consistent strategy**: options **must be chosen**
  - Other options are possible ;-)
Guiding example

- **Train component:** bogie
  - complex loadings
  - “classical” structures:
    - wheel, rail, wheel axle
    - 150 years of industrial progress
    - continuous improvements
  - steels: well known material

- **Illustration:**
  - **residual stresses** due to the manufacturing process (quenching)
  - **asymptotic regimes** under cyclic loadings (in particular shakedown)
  - **fatigue** design: see next part with Thierry
Wheel: material scale

Non linear kinematic hardening

\[ f(\sigma) = \sqrt{\frac{3}{2}(s - X) : (s - X) - \sigma_0} \]

\[ \dot{X} = \frac{2}{3} H \dot{\varepsilon}_p - \frac{2}{3} \gamma X \dot{p} \]

\[ \dot{p} = \sqrt{\frac{2}{3} \dot{\varepsilon}_p : \dot{\varepsilon}_p} \]

Parameters: calibrated on the stabilized cycles

Stabilized cycles

Available in ABAQUS

[Langueh et al., 2012]
Wheels: forging and quenching
Wheel: **structure scale**

- Structural computations:
  - From the previous step: **REV’s constitutive law = material**
  - From geometry+BC (+ loading) : **structure**
    - step 1: **residual stress** estimation (quenching: temperature loading)
    - step 2: **cyclic loading** due to contact (local stress paths, shakedown or not)
Residual stress simulations

Radial stresses

Plasticity with non linear kinematic hardening:
- between RT and 1000°C
- monotonic curves

[Brunel, 2007]

Maximum value in the flange:
293 Ma (in tension)

Circumferential stresses

Minimum value in the wheel tread:
- 242 MPa (in compression)

[Langueh et al., 2012]
Asymptotic responses under periodic loadings

- Structures submitted to periodic loadings: what are the possible asymptotic responses?
  - is the structure always elastic?
  - is there an asymptotic response when $t \to \infty$?
  - what is the asymptotic response when $t \to \infty$?

![Graphs showing elastic shakedown, plastic shakedown, and ratcheting](image-url)
Fatigue and shakedown

- Macroscopic plastic shakedown
- Mesoscopic plastic shakedown
- Mesoscopic elastic shakedown

Graph showing stress amplitude $\Delta\sigma$ vs. number of cycles $N_f$ with fatigue limit and endurance limits.
Shakedown state: steady-state algorithm

Dang Van and Maitournam (1993)

Eulerian model:

\[
\frac{dA}{dt} = \frac{\partial A}{\partial t} + v \cdot \text{grad}(A)
\]

- stationary assumption
- one streamline = one cycle

Dang Van’s diagram: see next part

Elastoplastic simulation

Dang Van et al., 2012
Proposed framework

Shakedown based framework in plasticity and in fatigue

Constitutive models?
Parameters calibration?
Algorithms?
Numerical implementation? (local)
Fatigue criterion? See next part
FE simulation? (global)
Outline

- **Material scale**
  - yield surface
  - strain hardening

- **Structure scale**
  - structural hardening
  - residual stresses

- **Numerical aspects**
  - Material scale: **radial-return**
  - Structure scale: **direct algorithms**

- Some illustrations, synthesis, remarks and references
Two scales

- Material scale
  - yield surface
  - strain hardening

- Structure scale
  - structural hardening
  - residual stresses

- Numerical aspects
  - Material scale: radial-return
  - Structure scale: direct algorithms

Constitutive models?
Parameters calibration?
Algorithms?

Some illustrations, synthesis, remarks and references
Plasticity: material scale

Material scale:
- question: **when does plasticity occur?**
  - from uniaxial to multiaxial plasticity: plastic criterion versus **yield stress**
- question: **how does plastic flow occur?**
  - flow rule: **hardening** concept

Structure scale:
- structural hardening
  - impact of stress/strain heterogeneities
- residual stresses
  - impact of strain incompatibilities
From experimental facts

Surface initiale de plasticité:

Eaux de traction-torsion (Bui, 1970)

Offset

Premières observations

σ₀ est dans le domaine d'élasticité.

Surface seuil de plasticité

Domaine d'élasticité

Uniaxial: 1D

Multiaxial: 3D

Cyclic
From experimental facts

Uniaxial: 1D
Chapitre 1
Introduction au comportement inélastique des matériaux et des structures

1.1 Les comportements macroscopiques des matériaux

Pour identifier le comportement macroscopique d'un matériau, on commence par réaliser des essais uniaxiaux sur éprouvette, i.e. des essais où l'on exerce des forces de traction ou de compression dans une direction - soit $e_1$). Si l'éprouvette est cylindrique et si la réponse est homogène, alors le tenseur des contraintes sera de la forme $\sigma = e_1 e_1$, ce qui justifie...

Tin-Silver-Copper alloy
$T^\circ C = 20^\circ C$

[Dompierre et al., 2011]
Perfect plasticity:

- Yield stress: \( f(\sigma) = |\sigma| - \sigma_0 \)
- Strain additive decomposition:
  \[ \varepsilon = \varepsilon^e + \varepsilon^p \]
- Elasticity:
  \[ \sigma = E(\varepsilon - \varepsilon^p) \]
- Yield stress and flow rule:
  \[ |\sigma| \leq \sigma_0, \begin{cases} 
  \dot{\varepsilon}^p \geq 0 & \text{si} \quad \sigma = +\sigma_0 \\
  \dot{\varepsilon}^p = 0 & \text{si} \quad |\sigma| < \sigma_0 \\
  \dot{\varepsilon}^p \leq 0 & \text{si} \quad \sigma = -\sigma_0 
\end{cases} \]
- \( \dot{\varepsilon}^p \) du signe de \( \sigma_0 \)
  \[ \dot{\varepsilon}^p = \dot{\lambda} \, \text{signe}(\sigma_0) \quad \text{avec} \quad \dot{\lambda} \geq 0 \]
From experimental facts

Surface initiale de plasticité: essais de traction-torsion (Bui, 1970)

Surface seuil de plasticité:

Domaine d'élasticité: $\sigma_0$ est dans le domaine d'élasticité.

Surf ace seuil ellipse:

Uniaxial: 1D

Multiaxial: 3D
From 1D to 3D plasticity

‣ What about a generalization?
  ‣ from yield stress to **plastic/yield criterion** and/or **yield surface**:

\[
f(\bar{\sigma}) = |\sigma| - \sigma_0 \quad \longrightarrow \quad f(\bar{\sigma}) \leq 0
\]

‣ History: multiaxial test in **tension-torsion**
From 1D to 3D plasticity: yield stress

- What about a generalization?
  - from yield stress to **plastic/yield criterion** and/or **yield surface**:
    \[
    f(\sigma) = \|\sigma\| - \sigma_0 \rightarrow f(\sigma) \leq 0
    \]

- History: multiaxial test in **tension-torsion**

\[
\sigma = \begin{bmatrix}
\sigma & \tau & 0 \\
\tau & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
F
\]

\[
C
\]
Tension-torsion tests

- Yield stress determination in pure tension

\[ \sigma = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

Yield stress in pure tension

Offset: $10^{-5}$ (10^{-3} \%)
Tension-torsion tests

- Yield stress determination in pure torsion

\[
\sigma = \begin{bmatrix}
\sigma & \tau & 0 \\
\tau & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Limite d’élasticité en torsion pure

Yield stress in pure tension

\[
\frac{\tau_0}{\mu}
\]

\[
\sigma_0 \quad \sigma
\]
Yield stress determination for radial loadings

\[ \sigma = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

Tension-torsion tests
Yield surface in tension-torsion

\[ \sigma = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
Taylor’s experiments (1931)

[Taylor and Quinney, 1931]
Bui’s experiments (1969)
3D plasticity: yield stress

- physical mechanisms: **plastic slip** (crystal planes, dislocations)

- plastic deformation **without volume change**
- activated with **shear stress**

- **What about yield criteria:** two main theories
  - criterion based on **maximal shear stress** (Tresca, 1874)
  - criterion based on **maximum distortion energy** (von Mises, 1913)

- no influence of the hydrostatic pressure (no volume change)
von Mises plastic criterion

- **Maximum distorsion energy criterion:**
  - based on **deviatoric stress**,
    \[ \bar{\sigma} = \bar{s} + \frac{1}{3} \text{tr}(\bar{\sigma}) \]
  - isotropic function of the stress tensor = function of its **invariants**
  - function of the **second invariant** of the deviator:
    \[ J_2 = \frac{1}{2} \bar{\sigma} : \bar{\sigma} \]

- **von Mises criterion** [1913]
  \[ f(\bar{\sigma}) = \sqrt{\frac{3}{2} \bar{s} : \bar{s} - \sigma_0} \]
  \[ f(\bar{\sigma}) = \sqrt{\frac{3}{2} \bar{\sigma} : \bar{\sigma} - \frac{1}{2} \text{tr}(\bar{\sigma})^2 - \sigma_0} \]
Surface seuil de plasticité

Premières observations: $\sigma_0$ est dans le domaine d'élasticité. Le domaine d'élasticité est convexe.

Surf ace seuil ellipse:

- Grand axe
- Grand axe

$$f(\sigma) = \sqrt{\sigma + 3\tau^2} - \sigma_0$$

$\sigma_0$ est dans le domaine d'élasticité. Le domaine d'élasticité est convexe.

von Mises in tension-torsion

$$\sigma = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
From 1D to 3D plasticity: flow rule

- 3D generalization?
  - Total strain additive decomposition: \( \varepsilon = \varepsilon^e + \varepsilon^p \)
  - Plastic deformation **without volume change**:
    \[
    tr(\varepsilon^p) = 0
    \]
  - Yield surface:
    \[
    f(\sigma) \leq 0
    \]
  - And when \( f(\sigma) = 0 \)? What about \( \dot{\varepsilon}^p \)?

\[
\begin{align*}
\text{uniaxial } |\sigma| \leq \sigma_0, \quad & \begin{cases}
\dot{\varepsilon}^p \geq 0 & \text{si } \sigma = +\sigma_0 \\
\dot{\varepsilon}^p = 0 & \text{si } |\sigma| < \sigma_0 \\
\dot{\varepsilon}^p \leq 0 & \text{si } \sigma = -\sigma_0
\end{cases} \\
\dot{\varepsilon}^p &= \dot{\lambda} \text{signe}(\sigma_0) \\
\text{avec } & \dot{\lambda} \geq 0
\end{align*}
\]
Tension-torsion tests: strain measurements

- Strain gauges:
  \[ \varepsilon \]

- Applied force and torque:
  \[ \sigma = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
  \[ \varepsilon^e = C^{-1} : \sigma \]

- Plastic strain deduced from total and elastic strain tensors:
  \[ \varepsilon^p = \varepsilon - \varepsilon^e \]
  \[ \dot{\varepsilon}^p = \dot{\varepsilon} - \dot{\varepsilon}^e = ||\dot{\varepsilon}^p||n \]
Tension-torsion tests: normality law

\[ \dot{\varepsilon}^p = \dot{\varepsilon} - \dot{\varepsilon}^e = |\dot{\varepsilon}^p| n \]

[Bui, 1969]
3D plasticity: flow rules

- **Generalization?**
  - Strain additive decomposition: $\varepsilon = \varepsilon^e + \varepsilon^p$
  - Experimental observation: plastic deformation **without volume change**
    
    $$tr(\varepsilon^p) = 0$$
  - Yield surface: $f(\sigma) \leq 0$

- Flow normal to the yield surface: **collinear to the gradient of** $f$
  (normality rule)
  
  $$f(\sigma) = |\sigma| - \sigma_0$$
  
  $$\dot{\varepsilon}^p = \lambda \frac{\partial f}{\partial \sigma} = \lambda \sqrt{\frac{3}{2} \frac{s}{||s||}}$$
  von Mises

- Plastic multiplier:
  $$\dot{\lambda} \geq 0$$

- Compact equation:
  $$\dot{\lambda} f = 0$$
From experimental facts

Surface initiale de plasticité: essais de traction-torsion (Bui, 1970)

Surface seuil de plasticité

Domaine d'élasticité

Premières observations

Surface seuil ellipse:

Uniaxial: 1D

Multiaxial: 3D

Cyclic
ID plasticity: hardening

- What is **hardening**?
  - Deformability evolution of the material
  - **Strengthening of a metal** by plastic deformation: increasing stress is required to produce additional plastic strain

- Yield stress evolution
  - **Mechanisms**: dislocation movements, dislocation density evolution as barriers to plastic slip, …

- What is **cyclic** hardening?
Cyclic strain-controlled tests

- **Principle**: tension-compression with increasing strain amplitude

[Chaboche, 1980]
ID plasticity: cyclic hardening

- Cyclic hardening:

- Cyclic softening:
Cyclic hardening curve

- **Principle**: tension-compression with increasing strain amplitude
  - cyclic hardening tests
  - strong assumption: search for the stabilized response of the material in order to calibrate the parameters (remember: shakedown)
Cyclic hardening curve

- **Cyclic softening:**
  - cyclic hardening curve below the monotonic one (see figure)

- **Cyclic hardening:**
  - cyclic hardening curve above the monotonic one

- Depends on:
  - hardening **mechanisms**
    (dislocations density, precipitates, metallic inclusions, …)
  - manufacturing **process**, material residual state, …

[Hosford, 2005]
3D plasticity: hardening
3D plasticity: hardening

- How to generalize?
  - Evolution of the **yield surface**
  - Size variations: **isotropic case**
  - Movements in the stress space: **kinematic case**
  - Distorsion: delicate modeling problem (see *ratcheting*)

- Hardening
  - Isotropic case: **scalar**
  - Kinematic case: **tensor**
Yield surface evolution in tension-torsion

\[ f(\sigma) = \sqrt{\sigma + 3\tau^2} - \sigma_0 \]
$f(\sigma) = \sqrt{\sigma + 3\tau^2} - \sigma_0$
\[ f(\sigma) = \sqrt{\sigma + 3\tau^2} - \sigma_0 \]
\( f(\underline{\sigma}) = 0 \) stay on the yield surface

\( f(\underline{\sigma}) = 0 \iff \) stay on the yield surface
\[ \sigma f(\sigma) = \sqrt{\sigma + 3\tau^2} - (\sigma_0 + R) \sigma \]
same curves

kinematic

isotropic
Kinematic hardening

- **Definition**: translation of the elastic domain

\[ S = S_0 + X \]

- One has to precise the evolution of \( X \)

- **Example**: linear kinematic hardening (Prager)

\[ X = H \varepsilon_p = \frac{2}{3} H \varepsilon_p \]

---

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( \varepsilon )</th>
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<tbody>
<tr>
<td>( \sigma_0 )</td>
<td>( E )</td>
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\[ H_{tan} = \frac{EH}{E + H} \]

\[ |\sigma - H \varepsilon_p| \leq \sigma_0 \]

---

W. Prager (1903-1980)

Baushinger effect
Kinematic hardening

- Non linear kinematic hardening (Armstrong-Frederick, Chaboche)

\[ f(\sigma) = \sqrt{\frac{3}{2}(s - X) : (\dot{s} - \dot{X}) - \sigma_0} \]

\[ \dot{X} = \frac{2}{3} H \dot{\varepsilon}_p - \frac{2}{3} \gamma X \dot{\varepsilon}_p \]

\[ \dot{\varepsilon}_p = \sqrt{\frac{2}{3} \dot{\varepsilon}_p : \dot{\varepsilon}_p} \]

Parameters calibration on cyclic hardening curve
Isotropic hardening

- **Definition**: dilatation of the elastic domain

\[ S = R \cdot S_0 \]

- One has to precise the evolution of the yield stress: \( R \)

- **Example**: von Mises with isotropic hardening

\[ f(\sigma) = \sqrt{\frac{3}{2} \sigma : \sigma - \frac{1}{2} tr(\sigma)^2 - \sigma_0 - R(p)} \]

\( p \) cumulated plastic strain

\[ \dot{p} = \sqrt{\frac{2}{3} \dot{\varepsilon}_p : \dot{\varepsilon}_p} \]

\[ p = \int_0^t \dot{p} \, dt \]
Isotropic hardening

- Example: von Mises with isotropic hardening

\[ f(\sigma) = \sqrt{\frac{3}{2} \sigma : \sigma - \frac{1}{2} tr(\sigma)^2 - \sigma_0 - R(p)} \]

\[ \dot{p} = \sqrt{\frac{2}{3} \dot{\varepsilon}_p : \dot{\varepsilon}_p} \]

- Linear isotropic hardening: \( R(p) = H \cdot p \)

- Non-linear isotropic hardening:

\[ R(p) = R_\infty (1 - \exp(-bp)) \]
Phenomenological approach

- Combined hardening:
  - non linear isotropic hardening: \( R(p) = R_\infty (1 - \exp(-bp)) \)
  - non linear kinematic hardening: \( \dot{X} = \frac{2}{3} H \ddot{\varepsilon}_p - \frac{2}{3} \gamma X \dot{p} \)
  - Back-stress, hardening saturation, recovering, …

- Example: ratcheting modeling

[Portier et al., 2000]
Combined hardening laws

Non-linear kinematic hardenings (Chaboche)

Constitutive equations of the NLK model

Strain decomposition: \( \varepsilon = \varepsilon' + \varepsilon'' \)

Hooke’s law: \( \varepsilon' = \frac{1}{2a} \left( 1 - \sqrt{1 - \frac{1}{4} \sigma + \frac{1}{4} } \right) \cdot \sigma \) with \( \mu = \frac{E}{2(1+\nu)} \)

Yield function: \( f(\sigma, R, \dot{X}) = J_2(\sigma - \dot{X}) - R - k \)

where \( J_2 = \frac{1}{2} \left( S - \dot{X} \right) : (S - \dot{X}) \) and \( S = \sigma - \frac{1}{2} \nu \sigma \)

Flow rule: \( \dot{\varepsilon}' = J \frac{\dot{X}}{\sigma} = \dot{\lambda} g \) with \( \dot{\lambda} = \frac{1}{\sigma} \frac{S - \dot{X}}{S - \dot{X}} \)

Kinematic hardening rule: \( \dot{X} = X_1 + X_2 \)

with \( \dot{\varphi}(p) = \varphi(1 + (1 - \varphi) e^{-\alpha p}) \)

Isotropic hardening rule: \( \dot{R} = b(Q_w - R) \rho \)

with \( k, Q_w, b, \varphi, \omega, C_1, C_2, \gamma_1, \gamma_2 \) material parameters

Non-linear kinematic hardenings (Ohno-Wang)

Constitutive equations of the OW model

Strain decomposition: \( \varepsilon = \varepsilon' + \varepsilon'' \)

Hooke’s law: \( \varepsilon' = \frac{1}{2a} \left( 1 - \frac{\sigma}{\varphi_0} + \frac{1}{2} \sigma \right) \cdot \sigma \) with \( \mu = \frac{E}{2(1+\nu)} \)

Yield function: \( f(\sigma, R, \dot{X}) = J_2(\sigma - \dot{X}) - R - k \)

where \( J_2 = \frac{1}{2} \left( S - \dot{X} \right) : (S - \dot{X}) \) and \( S = \sigma - \frac{1}{2} \nu \sigma \)

Flow rule: \( \dot{\varepsilon}' = J \frac{\dot{X}}{\sigma} = \dot{\lambda} g \) with \( \dot{\lambda} = \frac{1}{\sigma} \frac{S - \dot{X}}{S - \dot{X}} \)

and \( \dot{J} = \frac{\gamma_{\varphi}(p)}{\gamma_{\varphi}(p)} \dot{X}_i = \frac{3}{2} \left( \frac{X_i}{X_i} \right) \cdot k_i = \frac{X_i}{X_i} \)

Kinematic hardening rule: \( \dot{X} = X_1 + X_2 \)

with \( \dot{\varphi}(p) = \varphi(1 + (1 - \varphi) e^{-\alpha p}) \)

where \( \langle u \rangle \) are the Max Cauchy brackets: \( \langle u \rangle > 0 \) if \( u > 0 \) and \( \langle u \rangle = 0 \) if \( u < 0 \)

Isotropic hardening rule: \( \dot{R} = b(Q_w - R) \rho \)

with \( k, Q_w, b, \varphi, \omega, C_1, C_2, \gamma_1, \gamma_2, m_1, m_2 \) material parameters

+ Burlet-Cailletaud + Tanaka

[Portier et al., Ratchetting under tension-torsion loadings: experiments and modelling, IJP, 2000]
Influence of hardening rule on ratcheting

Non-linear kinematic hardenings (Chaboche)  
Non-linear kinematic hardenings (Ohno-Wang)

Tension-torsion ratcheting tests  
Tension-torsion ratcheting tests

See also works of Calloch, Vincent, … Yield surface distortion, …

[Portier et al., 2000]
Some theoretical justifications

- From experiments (see Taylor, Bui for example):
  - **convexity** yield surface
  - **normality** law

- **Drücker-Illyushin** postulate:
  - The deformation work has to be positive in the case of a *strain cycle* respecting the evolution law.

\[ \Delta W = \int_{t_0}^{t_1} \sigma : \dot{\varepsilon} dt \geq 0 \]

- Strain cycle: \( \varepsilon(t_1) = \varepsilon(t_0) \)
Some theoretical justifications

- **Maximal dissipation principle** (Hill, *demonstration in Marigo for ex.*)
  - Inequality deduced from Drücker-Illushin postulate in perfect plasticity:
    
    $$(\sigma(t) - \sigma^*) : \dot{\varepsilon}^P(t) \geq 0 \quad \forall \sigma^* \in S$$

  - Implies:
    - **convexity** of the yield surface
    - **normality** law

  [Hill, 1950]

  ![Diagram with yield surface and stress-strain relationship](image)
Thermodynamics of Irreversible Processes


- **Standard Generalized Materials (Halphen and Nguyen, 1975)**
  - **free energy potential**: state equations
    \[ w(\varepsilon, \varepsilon^p, \alpha) = w_e(\varepsilon - \varepsilon^p) + w_\alpha(\alpha) \]
  - **dissipation potential**: complementary equations
    \[ \varphi(\dot{\varepsilon}^p, \dot{\alpha}, \varepsilon, \varepsilon^p, \alpha) \quad \text{and} \quad (A_{\varepsilon^p}, A_\alpha) \in \partial(\dot{\varepsilon}^p, \dot{\alpha}) \varphi(\dot{\varepsilon}^p, \dot{\alpha}; \varepsilon, \varepsilon^p, \alpha) \]
  - **dual dissipation potential**: Legendre-Fenchel transform
    \[
    \varphi^*(A_{\varepsilon^p}, A_\alpha) = \sup_{\dot{\varepsilon}^p, \dot{\alpha}} \left\{ \frac{A_{\varepsilon^p}}{\dot{\varepsilon}^p} + \frac{A_\alpha}{\dot{\alpha}} : \varphi(\dot{\varepsilon}^p, \dot{\alpha}) \right\}
    \]
    \[ (\dot{\varepsilon}^p, \dot{\alpha}) \in \partial(A_{\varepsilon^p}, A_\alpha) \varphi^*(A_{\varepsilon^p}, A_\alpha) \]
Potential, dual potential and sub-differential

Example: uniaxial perfect plasticity \( \varphi(\dot{\varepsilon}^p) = \sigma_0|\dot{\varepsilon}^p| \)

\( \varphi(\dot{\varepsilon}^p) \)

\( \varphi^*(A) = \sup_{\dot{\varepsilon}^p} \{\sigma\dot{\varepsilon}^p - \sigma_0|\dot{\varepsilon}^p|\} = \sup_{|\dot{\varepsilon}^p|} \{|\sigma| - \sigma_0|\dot{\varepsilon}^p|\} \)

\( \varphi^*(A) \)

\( A \in \partial_{\dot{\varepsilon}^p} \varphi(\dot{\varepsilon}^p) \)

\( \dot{\varepsilon}^p \in \partial_A \varphi^*(A) \)

\( A = \sigma \)
Some equivalences

- **Dual potential:**
  - indicator of a convex set:
    \[ |\sigma| - \sigma_0 \leq 0 \]
  - sub-gradient:
    \[ \dot{\varepsilon}^P \text{ du signe de } \sigma_0 \]

- **Equivalence with:**
  - convex yield surface:
    \[ f(\sigma) = |\sigma| - \sigma_0 \]
  - normality law
    \[ \dot{\varepsilon}^P = \dot{\lambda} \text{ signe}(\sigma_0) \quad \dot{\lambda} \geq 0 \]

Then, what is the interest of TPI? ;-)
Positive dissipation?
(Only?) interest of thermodynamics: the energy balance and the heat coupled equation

- The energy balance and the heat coupled equation
  - The energy balance:
    \[
    \sigma : \dot{\varepsilon} = \rho(\dot{e} - T\dot{s}) + D_1 = \rho(\dot{w} + s\dot{T}) + D_1
    \]
    Provided « work » = Recoverable « work » + Intrinsic dissipation

- The heat coupled equation:
  \[
  \rho \mathcal{C}_\varepsilon \dot{T} + div(q) = r + \sigma : \dot{\varepsilon}^p - A_{\alpha} : \dot{\alpha} + \left( T \frac{\partial \sigma}{\partial T} : \dot{\varepsilon}^e + T \frac{\partial A}{\partial T} : \dot{\alpha} \right)
  \]
  - external heat supply (volume / surface)
  - intrinsic dissipation
  - reversible thermomechanical couplings
**« Classical » thermoelastoplasticity**

- The heat coupled equation:

\[
\rho C_v \dot{T} + \text{div}(q) = \underbrace{\sigma : \dot{\varepsilon}^p - A_{\alpha} : \dot{\alpha}}_{D_1} - T \frac{E\alpha}{1 - 2\nu} \text{tr}(\dot{\varepsilon}^e)
\]

- no external heat supply in volume
- no coupling between hardening and temperature (no phase transformation for ex.)

- a strong restriction for models:
  - kinematic hardening: \( D_1 = \sigma : \dot{\varepsilon}^p - X : \dot{\alpha} \)
  - isotropic hardening: \( D_1 = \sigma : \dot{\varepsilon}^p - R\dot{p} \)
Towards temperature measurements for thermomechanical sources determination (« à la Chrysochoos », see later)
Plasticity: structure scale

- **Material scale:**
  - question: when does plasticity occur?
    - from uniaxial to multiaxial plasticity: plastic criterion versus yield stress
  - question: how does plastic flow occur?
    - flow rule: hardening concept

- **Structure scale:**
  - structural hardening
    - impact of stress/strain heterogeneities
  - residual stresses
    - impact of strain incompatibilities
What is structural hardening?

- Example of a three bars truss
  - Bars only in tension-compression, section A
  - Equilibrium:
    \[ \sum_{i=1}^{3} N_i t_i = F \]
  - Yield criterion: \[ f(\sigma) = |\sigma| - \sigma_0 \]
    \[ |N_i| \leq N_0 \]

- Elastic solution:
  - \[ F = \frac{AE\delta}{l}(1 + 2\cos^3\alpha) \text{ for } \delta \leq \frac{N_0 l}{AE} \]
  - [Leckie et al., 1990]
Plasticity of the three-bars truss

- Plasticity of central bar:
  
  \[ F = N_0 + \frac{2AE\delta}{l} \cos^3 \alpha \text{ for} \]
  
  \[ \frac{N_0 l}{AE} \leq \delta \leq \frac{N_0 l}{AE \cos^2 \alpha} \]

- Limit load:
  
  \[ F = F_{\text{lim}} = N_0 (1 + 2 \cos \alpha) \text{ for} \]
  
  \[ \delta \geq \frac{N_0 l}{AE \cos^2 \alpha} \]

- Despite local perfect plasticity, **structural hardening** due to strain incompatibilities and residual stresses

[Leckie et al., 1990]
Generalization

- Infinite number of bars:

- Tube in torsion:

[Marigo, 2017]

[Lubliner, 2008]
Generalization

- Infinite number of bars:

- Main conclusions:
  - Structural hardening = **kinematic hardening**
  - Forces in bars: **localization** of the global forces and **residual stresses**

\[ \sigma = \frac{A}{\Sigma} : \Sigma + \sigma^{res} \]

- See homogenization of polycrystals for example
Residual stresses

- Residual stresses in bars:
  - \( \sigma_v = -2 \frac{N_0}{A} \frac{\sin^2 \alpha \cos \alpha}{1 + 2 \cos^3 \alpha} \) for the vertical bar (compression),
  - \( \sigma_i = \frac{N_0}{A} \frac{\sin^2 \alpha}{1 + 2 \cos^3 \alpha} \) for the inclined bar (tension)

- Generalization:
  - **self-equilibrated** stress field:
    \[ \text{div} \sigma^{res} = 0 \text{ sur } \Omega \text{ and } \sigma^{res} \cdot n = 0 \text{ on } S_T \]
  - one can (easily) prove that:
    \[ \int_{\Omega} \text{tr}(\sigma) d\Omega = 0 \]

- some zones in tension + some zones in compression
Real and elastic solutions

The **real solution** is composed with:
- a stress field $\sigma$ statically admissible and plastically admissible (i.e. $f(\sigma) \leq 0$)
- a strain field $\varepsilon$ kinematically admissible and a plastic strain field $\varepsilon^p$
- as: $\sigma = C : (\varepsilon - \varepsilon^p)$

The **pure elastic solution** is composed with:
- a stress field $\sigma^{el}$ statically admissible
- a strain field $\varepsilon^{el}$ kinematically admissible
- as: $\sigma^{el} = C : \varepsilon^{el}$
Residual solution

- The **residual solution** is defined by the difference between the real solution and the elastic one:

\[
\begin{align*}
\sigma^{res}(M, t) &= \sigma(M, t) - \sigma^{el}(M, t) \\
\varepsilon^{res}(M, t) &= \varepsilon(M, t) - \varepsilon^{el}(M, t)
\end{align*}
\]

Therefore:

- $\sigma^{res}$ is SA0 and $\varepsilon^{res}$ is KA0

- $\sigma^{res} = C : (\varepsilon^{res} - \varepsilon^p)$ as $\sigma^{res} = \sigma - \sigma^{el}$

$\sigma^{res} = C : (\varepsilon - \varepsilon^p) - C : \varepsilon^{el}$

$\sigma^{res} = C : (\varepsilon^{res} - \varepsilon^p)$

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Residual stresses

- Therefore:
  - \( \text{div} \sigma_{\text{res}} = 0 \) on \( \Omega \)
  - \( \sigma_{\text{res}} \cdot n = 0 \) on \( S_T \) [SA0]
  - \( \sigma_{\text{res}} = C : (\varepsilon_{\text{res}} - \varepsilon^p) \)

- \( \varepsilon_{\text{res}} = \frac{1}{2} (\text{grad}u_{\text{res}} + T \text{grad}u_{\text{res}}) \) [KA0]
- \( u_{\text{res}} = 0 \) on \( S_u \)

- The residual stress field is solution of an elastic problem with an initial strain field \( \varepsilon^p \) and zero forces.

- To compute residual stress field:
  \[
  \sigma_{\text{res}}(M, t) = \sigma(M, t) - \sigma_{\text{el}}(M, t)
  \]
  
  Elastic unloading
Incompatibilities

- If $\varepsilon^p$ is compatible:
  - then it exists $u^p$ as: $\varepsilon^p = \frac{1}{2}(\text{grad}u^p + T\text{grad}u^p)$
  - $u^p = 0$ on $S_u$

  - then $u^{res} = u^p$ (uniqueness of the elastic solution) and $\sigma^{res} = 0$

- If $\varepsilon^p$ is not compatible:
  - then, there exists $\varepsilon^{el}_{res}$ as: $\varepsilon^r_{res} = \varepsilon^{el}_{res} + \varepsilon^p$
  - in order to obtain $\varepsilon^r_{res}$ \( \Xi \)KA0

Geometrically

Non integrability
Illustration: watermelon and rhubarb
Two scales

- Material scale
  - yield surface
  - strain hardening

- Structure scale
  - structural hardening
  - residual stresses

- Numerical aspects
  - Material scale: radial-return
  - Structure scale: direct algorithms

Some illustrations, synthesis, remarks and references
Radial-return algorithm

\[ f(n+1) = 0 \]

\[ f(n) = 0 \]

\[ f(\sigma_{n+1}) > 0 \]

\[ f(\sigma_n) = 0 \]
3D plasticity: numerical implementation

- Numerical resolution: **determination of the plastic strains**
  \[ \varepsilon^p : \text{internal variable} \]

  - At each time increment, the mechanical state at \( t_{n+1} \):
    \[
    S_{n+1} = \{ u_{n+1}, \varepsilon_{n+1}^p, \varepsilon_{n+1}^p, \sigma_{n+1} \}
    \]
  - has to be calculated by knowing:
    \[
    S_n = \{ u_n, \varepsilon_n, \varepsilon_n^p, \sigma_n \} \quad \text{at} \quad t_n
    \]
  - and the loading:
    \[
    (f_{n+1}, u^{D}_{n+1}, T^{D}_{n+1}) \quad \text{at} \quad t_{n+1}
    \]

- **Two particular aspects:**
  - **local resolution scheme**
  - Newton algorithm: **tangent operator** (in order to build the tangent matrix)
3D plasticity: numerical implementation

- **Local resolution:** determination of the *plastic strains* at the time increment \( (n+1) \)
  - generally: backward Euler algorithm
    \[
    \varepsilon_{n+1}^p = \varepsilon_n^p + \Delta t F(\varepsilon_{n+1}^p)
    \]
  - non linear problem: *Newton(-Raphson)* algorithm

- **Global resolution:** determination of the *tangent operator* for the creation of the *tangent matrix*
  - Stress at time increment \( (n+1) \):
    \[
    \sigma_{n+1} = \sigma_n + C(\Delta \varepsilon - \Delta \varepsilon^p)
    \]
  - **Tangent operator** :
    \[
    \frac{\partial \sigma_{n+1}}{\partial \Delta \varepsilon} = C : \left( I - \frac{\partial \Delta \varepsilon^p}{\partial \Delta \varepsilon} \right)
    \]
    local “stiffness”
3D plasticity: radial-return algorithm

- Yield function and normality rule:
  \[ f(\sigma) \leq 0 \quad \dot{\varepsilon}^p = \lambda \frac{\partial f}{\partial \sigma} \]

- Trial elastic stress:
  \[ \sigma^{elas}_{n+1} = \sigma_n + C : \Delta \varepsilon \]

- For the solution, let the normal to the convex be:
  \[ N_{n+1} = \frac{s_{n+1}}{\| s_{n+1} \|} \]

\[ s^{elas}_{n+1} = \left( \| s_{n+1} \| + 2 \mu \Delta p \sqrt{\frac{3}{2}} \right) N_{n+1} \]

- Solution \( \sigma_{n+1} \): radial projection of the trial stress on the convex at \( n+1 \)

Moreau [1971]
Nguyen [1977]
Simo et al. [1985]
Direct algorithms
Some numerical considerations

- Asymptotic responses:
  - stabilization for elastic / plastic shakedown
  - fatigue criterion: analysis of these stabilized response

- Can we calculate **directly** these asymptotic responses? Yes!
  - direct methods:
    - Direct Cyclic Algorithm
    - Steady-State Algorithm
    - Zarka’s method
  - based on the periodicity (stress, strain) of the response: loading Fourier decomposition, shakedown-based analysis, …
Direct Cyclic Algorithm

- **Principle:**
  - **Fourier** representation of the displacements, loadings, ... with n terms
  - **Global iteration:** Evaluation of the residual vector in m time points
  - Fourier representation of the residual vector with n coefficients
  - **Modified Newton iterations** to determine corrections to the displacements Fourier terms
  - **Local iteration** (see radial-return)
  - Enforcement of the **periodicity condition** on the solution (periodic plastic strains and stresses, residual vector tend to zero)

\[ R_0, F_0^c, F_0^A \]
\[ Q_0 = Q_0^c = Q_0^A = 0 \]

\[ KU_0 = F_0 + Q_0 \]
\[ KU_n^c = F_n^c + Q_n^c \]
\[ KU_n^A = F_n^A + Q_n^A \]

\[ e_k, \sigma_k^{SA} \]

\[ f(\sigma_k^{SA}) < 0 \]

[Maouche, 1997]
Some comparisons

In-phase biaxial cyclic loading

Out-of-phase biaxial cyclic loading

[Baudoin, 2016]
Some comparisons

[Pommier, 2003]
Two scales

- Material scale
  - yield surface
  - strain hardening

- Structure scale
  - structural hardening
  - residual stresses

- Numerical aspects
  - Material scale: radial-return
  - Structure scale: direct algorithms

Constitutive models?
Parameters calibration?
Algorithms?

Some illustrations, synthesis, remarks and references
Self-heating tests with **different thermoelastoplastic models**

\[ \rho C_e \dot{T} + \text{div}(q) = \sigma : \varepsilon^p - \frac{A}{E \alpha} : \dot{\varepsilon} - T \frac{E \alpha}{1 - 2\nu} \text{tr}(\dot{\varepsilon}^e) \]
Self-heating tests with **different thermoelastoplastic models**

[Vincent, 2008]
Isotropic vs. Kinematic hardening
Plate with circular hole

- In the line containing A, displacement-controlled loading-unloading
- Stress-Strain evolutions in B and C:
  - isotropic hardening
  - kinematic hardening
Isotropic hardening
Kinematic hardening
Application to an automotive piston
Idealistic mechanical characterization

- Linear hardening plasticity
Influence of the hardening law

Fatigue lifetime?
Structural effect on energy balance

- Thermoplasticity on structure with \textit{isotropic} or \textit{kinematic} hardening

![Diagram of Cook's membrane: (a) the geometry and (b) the mesh.]

Fig. 9. Cook’s membrane: (a) the geometry and (b) the mesh.

[Häkansson et al., 2005]
Energy balance

- **Isotropic hardening**

Fig. 11. Cyclic thermoelasto-plastic response for the isotropic hardening model. (a) Mechanical response; (b) temperature evolution versus time, solid line corresponds to point A and the dashed one to point B, cf. Fig. 9(b).

[Häkansson et al., 2005]
Energy balance

- **Kinematic** hardening

Fig. 10. Cyclic thermoclastic-plastic response for the kinematic hardening model. (a) Mechanical response; (b) temperature evolution versus time, solid line corresponds to point A and the dashed one to point B, cf. Fig. 9(b). [Häkansson et al., 2005]
Residual strain and deformation

- Exhaust manifolds under thermal shock: *residual deformation* with 4 different viscoplastic laws

[Szmytka, 2007]
Residual strain and deformation

- Exhaust manifolds under thermal shock: **residual deformation** with 4 different viscoelastic laws

[Diagram showing residual deformation over cycles with different viscoelastic laws]
Structure and material scales: coupling

- Contact pressure:
  - maximum shear stress under the surface

- Elastic-plastic simulations
  - first rolling cycles
  - asymptotic response: plastic shakedown?

- Delicate point: **shakedown**
  - highly depends on the plastic hardening constitutive law
  - highly depends on the loading evolution (contact pressure evolution due to plasticity!)

[Saint-Aimé, 2017]
Synthesis

Material scale:

- **Yield function** (example: von Mises): *when does plasticity occur?*
- **Flow rule** (example: normality rule): *how does plastic flow occur?*
  - without volume changes: deviator (counterexample: soil mechanics)
  - with or without **hardening**: evolution of the yield surface
    - **Isotropic**: size increases
    - **Kinematic**: yield surface translation in the deviator space
- Permanent strain: **plastic strain**
- Numerical implementation: **radial-return algorithm**

Structure scale:

- Incompatibilities: **structural hardening** and **residual stresses**
- **Direct computations** are possible (with more or less strong assumptions)
Remarks

- Elastoplastic constitutive laws:
  - **Combined hardening rules**: « classical » but influence on structure scale (ratcheting, energy balance, residual stress and deformation)
  - Parameters calibration: non uniqueness! Need a large cyclic database and/or … thermomechanical data for **energy balance**!
  - Many constitutive laws in **FE codes**, in other way UMat, Z-front, MFront, … (see references)

- **Extension to viscoplasticity**:
  - Viscoplastic potential regularizes the problem
  - Same type of hardening laws including **recovering**
Example:

Yield function:

\[ f = \sqrt{3J_2} - \sigma_y \]

Flow rule:

\[ \dot{\varepsilon}_{vp} = \sqrt{\frac{3}{2}} \langle \frac{J_2(\sigma - X) - \sigma_y}{\eta} \rangle^m \frac{\text{dev}(\sigma - X)}{J_2(\sigma - X)} \]

[Cailletaud et al., 2009]
Certainly more elements …

- Fabien Szmytka on cylinder heads,
- N. Saintier, L. Signor and C. Mareau on polycrystals in fatigue
- V. Maurel on crack growth in generalized plasticity
- S. Pommier on incremental crack growth
- P. Kanoute on stress gradient
- S. Fouvry on fretting fatigue
- …
Les travaux de recherche de Habibou Maitournam portent sur la modélisation des comportements thermomécaniques et asymptotiques des structures anélastiques sous chargements cycliques, la prévision de leur tenue à la fatigue ainsi que la modélisation de procédés. Ses travaux sont largement utilisés par les industries automobiles et ferroviaires. Il est professeur à l’ENSTA ParisTech et professeur associé à l’École polytechnique.

Cet ouvrage s’adresse principalement aux élèves des grandes écoles scientifiques ainsi qu’aux étudiants des universités suivant une voie spécialisée en mécanique des matériaux et des structures. L’objectif de cet ouvrage est de donner tous les éléments théoriques nécessaires à la mise en œuvre d’une démarche de détermination de la durée de vie des structures sous chargement cyclique (thermodynamique, comportement cyclique, théorie de l’adaptation et fatigue). En effet, de plus en plus de structures mécaniques, qu’elles soient aéronautiques, automobiles ou ferroviaires, travaillent hors de leur domaine de comportement linéaire, leur dimensionnement optimal nécessite une bonne maîtrise de leurs états thermomécaniques issus de la fabrication et de l’évolution de ceux-ci sous des chargements complexes, de service ou accidentels. Le comportement anélastique des matériaux et des structures sous chargements transitoires et cycliques est étudié en vue de la compréhension des principaux modes de ruine. Il est illustré par des nombreux exemples.

Questions et thématiques abordées :
1. Que devient l’énergie fournie à un système ?
L’énergie dans tous ses états : énergétique et thermodynamique.
2. Comment construit-on des modèles de comportement anélastique des matériaux ?
Thermoélasticité, élastoplasticité, viscoplasticité.
3. Comment mettre en œuvre numériquement ces modèles pour calculer des structures ?
4. Que reste-t-il quand la sollicitation disparaît ?
Contraintes résiduelles : origine, méthodes de détermination.
5. Comment répond la structure sollicitée cycliquement ? S’adapte-t-elle ou « craque »-t-elle ?
Structures sous chargements cycliques : état asymptotique : adaptation, accommodation, rochet ; théorie de l’adaptation : applications aux poutres et aux structures tridimensionnelles.
6. En pratique, il arrive que la structure fatigue...
Introduction à la fatigue des structures.

C’est une vision unitaire du comportement des matériaux et des structures (comportement mécanique et fatigue) sous l’angle dissipatif qui est exposée. L’angle énergétique avec le rôle clé de la dissipation et le formalisme « standard généralisé » sont mis en valeur. Des mises en œuvre simples permettent une compréhension de la théorie et des exercices adaptés détaillent son application.
References

- **Classical and useful:**
  - **J. Lubliner**, Plasticity theory, Dover editions, 2008
    

- **Historical:**
Special thanks

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Some questions?